The Galvin property at successors of singulars

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Galvin's Theorem

In a paper by Baumgartner, Hajnal and Maté [1], the following theorem due to F. Galvin was published:

Theorem 1 (Galvin's Theorem)

Suppose that $\kappa^{<\kappa} = \kappa$. Then for every normal filter U over κ, and for any collection $\langle A_\alpha \mid \alpha < \kappa^+ \rangle \in [U]^{\kappa^+}$ consisting of κ^+-many sets, there is a subcollection $\langle A_i \mid i \in I \rangle$, of size κ (i.e. $I \in [\kappa^+]^\kappa$) such that $\bigcap_{i \in I} A_i \in U$.

In particular, if *GCH* holds and κ is a regular cardinal then from κ^+-many clubs, one can always extract κ-many for which the intersection is a club.
Let us put this combinatorical/saturation property into a definition:

Definition 2 (Galvin's Property)

Let $\mathcal{F}$ be a filter over κ and $\mu \leq \lambda$. Denote by $Gal(\mathcal{F}, \mu, \lambda)$ the following statement:

$$\forall \langle A_i \mid i < \lambda \rangle \in [\mathcal{F}]^\lambda . \exists I \in [\lambda]^\mu . \bigcap_{i \in I} A_i \in \mathcal{F}$$

Example 3

1. Galvin's Theorem $\equiv$ If $\kappa^{<\kappa} = \kappa$ the $Gal(U, \kappa, \kappa^+)$ holds for every normal U over κ.
2. If $\mu' \leq \mu \leq \lambda \leq \lambda'$ then $Gal(\mathcal{F}, \mu, \lambda) \Rightarrow Gal(\mathcal{F}, \mu', \lambda')$.
3. If (e.g.) $\mathcal{F}$ contains all the final segments and $\mu = cf(\kappa)$ then $\neg Gal(\mathcal{F}, \mu, \mu)$.
4. $\mathcal{F}$ is μ-complete $\iff$ for every $\mu' < \mu$, $Gal(\mathcal{F}, \mu', \mu')$.

Most of the work presented here is the results of two projects. The first is a joint project with **Alejandro Poveda** and **Shimon Garti** where we studied the Galvin property on filters and some applications of it, we were specially interested with the club filter. The second project, joint with **Moti Gitik**, where we were mostly focused on κ-complete ultrafilters.

Applications of the Galvin property

- **Density of old sets in Prikry extensions**[7],[2]: Let U be a κ-complete ultrafilter and $\mu \leq \lambda$. Then the following are equivalent:
  - $Gal(U, \mu, \lambda)$
  - Every set of ordinals $x \in V^{Prikry(U)}$ with $|x|^{V^{Prikry(U)}} = \lambda$ contains a set $y \in V$ with $|y|^V = \mu$.
- **Adding Cohens with Prikry**[5]: $Gal(U, \kappa, \lambda)$ implies that $Prikry(U)$ does not add λ-many mutually generic Cohen functions to κ.
- **Quotients of Prikry-type forcings**[4]: Some generalization of Galvin's property (which hold for normal filters) is used to prove that quotients of a forcing $\mathbb{P}$ are κ^+-cc in $V^{\mathbb{P}}$, where $\mathbb{P}$ can be the Magidor-Radin forcing, the Prikry forcing with $P-$points (and potentially other Prikry-type forcings).
- **Partition relations**[2]: For example, if there is a uniform ultrafilter such that $Gal(U, \kappa^+, \lambda)$ holds then $\binom{\lambda}{\kappa} \to \binom{\kappa^+}{\kappa}$.
- **Kurepa trees**[3]: If U is a $\kappa-$complete ultrafilter, such that $Cub_\kappa \subseteq U$ which concentrates on E^κ_μ for some $\mu < \kappa$, then there is no Slim S-Kurepa tree for every stationary $S \subseteq E^\kappa_\mu$.
- Some consistently new instances of $\lambda \to (\lambda, \omega + 1)$, relation to strong generating sequence of ultrafilters, and more...

How far can we push Galvin's Theorem?

We can either try to relax the assumption of Galvin's theorem $\kappa^{<\kappa} = \kappa$ or improve the consequent. Let us start with the latter,

Theorem 4 ([4])

Suppose that $\kappa^{<\kappa} = \kappa$. Then for every filter U which is Rudin-Keisler equivalent to a finite product of P-point filters, $Gal(U, \kappa, \kappa^+)$ holds.

The proof of this theorem can be adapted to work for filters of the form:
$U - lim_\alpha U_\alpha$, $U - lim_\alpha(U_\alpha - lim_\beta(U_{\alpha,\beta}))$,
$U - lim_\alpha(U_\alpha - lim_\beta(U_{\alpha,\beta} - lim_\gamma U_{\alpha,\beta,\gamma}))$,
$U - lim_\alpha(U_\alpha - lim_\beta(U_{\alpha,\beta} - lim_\gamma U_{\alpha,\beta,\gamma})...)$

Corollary 5

In $L[U]$, every κ-complete (even σ-complete) ultrafilter W satisfy $Gal(W, \kappa, \kappa^+)$.

Question

Is it consistent to have a filter/ultrafilter U which is not of the previous form for which $Gal(U, \kappa, \kappa^+)$ holds?

Non-Galvin filters and ultrafilters

Finding non-Galvin filters is relatively easy.

Definition 6

A family of subsets of κ, $\langle A_i \mid i < \lambda\rangle$ with the property that for every $I, J \in [\lambda]^{<\kappa}$, $I \cap J = \emptyset \Rightarrow (\bigcap_{i\in I} A_i) \cap (\bigcap_{j\in J} A_j^c) \neq \emptyset$ is called a κ-independent family of size λ,

κ-independent families of size 2^κ always exist given that $\kappa^{<\kappa} = \kappa$. Moreover, without this cardinal arithmetic assumptions, λ-many mutually generic Cohen functions over a regular κ form a $\kappa-$independent family.

Proposition 1

Let $\mathcal{F}$ be the κ-complete filter generated by a κ-independent family of size λ, then $\neg Gal(\mathcal{F}, \kappa, \lambda)$.

Question

Is there a ZFC-construction of a κ-complete filter $\mathcal{F}$ such that $Cub_\kappa \subseteq \mathcal{F}$ and $\neg Gal(\mathcal{F}, \kappa, \kappa^+)$?

Certainly, the existence of a κ-complete ultrafilter which is not Galvin requires large cardinals. The first construction is due to S. Garti, S. Shelah and B.[3], starting from a supercompact. Lately we obtained this from optimal assumptions:

Theorem 7 ([5])

Assume GCH.

1. *If κ is a measurable cardinal then there is a forcing extension where there is a κ-complete ultrafilter U such that $Cub_\kappa \cup \{reg_\kappa\} \subseteq U$ and $\neg Gal(U, \kappa, \kappa^+)$.*
2. *If $o(\kappa) = 2$, then there is a forcing extension where there is a κ-complete ultrafilter U such that $Cub_\kappa \cup \{sing_\kappa\} \subseteq U$ and $\neg Gal(U, \kappa, \kappa^+)$.*
3. *If $o(\kappa) = \kappa^{++}$ then there is a forcing extension where there is a κ-complete ultrafilter $Cub_\kappa \cup \{reg_\kappa\} \subseteq U$ such that $\neg Gal(U, \kappa, \kappa^{++})$*

The Abraham-Shelah model

Trying to relax the assumption $\kappa^{<\kappa} = \kappa$ in Gavin's theorem, we have the following consistency result by Abraham and Shelah.

Theorem 8 (Abraham-Shelah forcing)

Assume GCH, let κ be a regular cardinal, and $\kappa^+ < cf(\lambda) \leq \lambda$. Then there is a forcing extension by a κ-directed, cofinality preserving forcing notion such that $2^{\kappa^+} = \lambda$ and there is a sequence $\langle C_i \mid i < \lambda \rangle$ such that:

1. *C_i is a club at κ^+.*
2. *for every $I \in [\lambda]^{\kappa^+}$, $|\bigcap_{i \in I} C_i| < \kappa$.*

In particular, $\neg Gal(Cub_{\kappa^+}, \kappa^+, 2^{\kappa^+})$.

A natural question is what happens on inaccessible cardinals? of course, by Galvin's theorem, we should be interested in weakly inaccessible Cardinals.

Question

Is it consistent to have a weakly inaccessible cardinal κ such that $\neg Gal(Cub_\kappa, \kappa, \kappa^+)$?

There are some limiting results due to Garti (see [6])

At successors of singular cardinals

Our focus is on the second case which does not fall under Abraham-Shelah's Theorem: is it consistent to have $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$ for a singular κ? Again, by Galvin's theorem, this would require violating SCH.

Theorem 9 ([2])

Assume GCH and that E is a (κ, κ^{++})-extender. Then there is a forcing extension where $cf(\kappa) = \omega$ and $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

The idea is to iterate Abraham-Shelah's forcing on inaccessibles up to and including κ using an Easton support. This produces $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$. Using a Woodin-like argument, based on Y. Ben-Shalom (see [8]), one can argue that κ remains measurable after the iteration. Finally, singularize κ using Prikry/Magidor forcing. The key lemma is to prove that Prikry forcing does not destroy a witness for the failure of the Galvin property:

Proposition 2

A κ^+-cc forcing preserves a witness for $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

Assuming larger cardinals, we were able to get this failure to hold globally, for every successor of singular cardinal.

The strong negation at successor of singulars

The sequence of clubs $\langle C_i \mid i < \kappa^+ \rangle$ produced by the Abraham-Shelah forcing, witnesses a stronger failure of $Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$, indeed for any $I \in [\kappa^{++}]^{\kappa^+}$, $\bigcap_{i \in I} C_i$ is actually of size less than κ. Let us denote this by $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.
Interestingly, the previous argument does not work for the strong negation:

Proposition 3

In general κ^+-cc forcings do not preserve $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

Indeed, any forcing which adds a set of size κ which diagonalizes $(Cub_\kappa)^V$ (e.g. diagonalizing the club filter, Magidor forcing with $o(\kappa) = \kappa$) kills $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

Question

Is it consistent that $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$ holds at a successor of a singular cardinal?

Two opposite results for Prikry forcing

On one hand Prikry forcing does not add a set of cardinality κ which diagonalizes $(Cub_\kappa)^V$:

Theorem 10 ([2])

Let U be a normal ultrafilter over κ. Let $\langle c_n \mid n < \omega\rangle$ be V-generic Prikry sequence for U, and suppose that $A \in V[\langle c_n \mid n < \omega\rangle]$ diagonalizes $(Cub_\kappa)^V$. Then, there exists $\xi < \kappa$ such that $A \setminus \xi \subseteq \{c_n \mid n < \omega\}$. In particular, $|A \setminus \xi| \leq \aleph_0$.

On the other hand, just forcing a Prikry sequence is not enough:

Theorem 11 ([2])

Let $\mathcal{C}$ be a witness for the strong negation. Then there exists $\mathcal{D}$, such that:

1. *$\mathcal{D}$ is also a witness for the strong negation;*
2. *For every normal ultrafilter U over κ, forcing with $Prikry(U)$ yields a generic extension where $\mathcal{D}$ cease to be a witness.*

Finish line

Thank you for your attention!

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