Davies's theorem and the coloring number of graphs

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Advances in Set Theory
July 10–14 2022

Theorem. (Roy O. Davies, 1974) If CH holds, $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, then there are functions $g_n, h_n : \mathbb{R} \to \mathbb{R}$ that

$$F(x,y) = \sum_{n=0}^{\infty} g_n(x)h_n(y).$$

Davies' construction had the property that all sums are finite (on the RHS all but finitely many terms are zero).
He showed that this stronger form is equivalent to CH. (Considering $F(x, y) = e^{xy}$.)

(Shelah) The original form of Davies's theorem is both consistent and independent with non-CH.

A *graph* is a pair (V, X) where V is a set (vertices), X is a set consisting 2-element subsets of V (edges).

Definition. A graph (V, X) is *Davies* if the following holds: if $F : X \to \mathbb{R}$, then there are functions $g_v : \omega \to \mathbb{R}$ $(v \in V)$ such that

$$F(v, w) = \sum_{n=0}^{\infty} g_v(n) g_w(n) \quad (\{v, w\} \in X).$$

Davies: if CH holds, then $K_{c,c}$ is Davies.
or better:
K_{ω_1,ω_1} is Davies without any condition.

(Shelah, 1997) If $\mathrm{MA}(\sigma\text{-centered})$ holds, then $K_{c,c}$ is Davies.

(Shelah, 1997) $K_{c,c}$ is not Davies if c or more Cohen reals are added.

(Roslanowski–Shelah, in 'The yellow cake') $K_{c,c}$ is Davies does not imply $\mathrm{MA}(\sigma$-centered).

Defintion. (Erdős–Hajnal, 1966) The *coloring number of* (V, X), $\mathrm{Col}(V, X)$ is the least cardinal μ s.t. there is a well ordering $<$ of V, that each vertex is joined to $< \mu$ smaller (under $<$) vertices.

Notice that $\mathrm{Chr}(X) \leq \mathrm{Col}(X)$.

The following are equivalent:
(a) $\mathrm{Col}(V, X) > \mu^+$
(b) if $f : V \to [V]^\mu$, then there is $\{x, y\} \in X$, such that $x \notin f(y)$, $y \notin f(x)$.

(Shelah, 1975) If (V, X) is a graph, $\lambda = |V|$ is singular, μ is a cardinal, all subgraphs (W, Y) of X with $|W| < \lambda$ have $\mathrm{Col}(W, Y) \leq \mu$, then $\mathrm{Col}(V, X) \leq \mu$.

Let X be a bipartite graph on $A \cup B$, $|A| = \lambda$, $|B| = \lambda^+$, and each $x \in B$ is joined to μ vertices in A (λ, μ infinite cardinals). Then $\mathrm{Col}(X) > \mu$.

Theorem. If (V, X) is a graph, $|V| \leq c$, $\mathrm{Col}(X) \leq \omega_1$ then X is Davies.

Theorem. If $\mathrm{Chr}(X) > c$, then (V, X) is not Davies.

Proof Try $F(x, y) = -1$ $(\{x, y\} \in X)$. If functions $\{g_v : v \in V\}$ are given, as $\mathrm{Chr}(X) > c$, there are $v \neq w \in V$, with $g_v = g_w$, $\{v, w\} \in X$. Then $\sum_n g_v(n) g_w(n) = \sum_n g_v(n)^2 \geq 0$, contradiction.

Theorem. It is consistent that $c = \aleph_3$ and if (V, X) is a graph with $\mathrm{Col}(X) = |X| = \aleph_2$, then (V, X) is not Davies.

Theorem. If the existence of a supercompact cardinal is consistent then it is also consistent that $c = \aleph_3$ and every graph X with $\mathrm{Col}(X) > \omega_1$ and any size is not Davies.

Thank you for your patience!