Partition Properties and Cardinalities

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Assuming AD, many cardinals have infinite exponent partition properties.

We say $\kappa \to (\kappa)^\lambda$ if for all partitions $\mathcal{P}\colon [\kappa]^\lambda \to 2$, there is a homogeneous set $H \subseteq \kappa$ of size κ. That is, $\mathcal{P} \restriction [H]^\lambda$ is constant.

We say $f\colon \lambda \to \kappa$ is of the correct type if it is increasing, everywhere discontinuous, and of uniform cofinality ω.

We say $\kappa \overset{\text{c.u.b.}}{\longrightarrow} (\kappa)^\lambda$ if for all partitions $\mathcal{P}$ of the function from λ to κ of the correct type, there is a c.u.b. $C \subseteq \kappa$ which is homogeneous for $\mathcal{P}$.

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Fact

The two forms of the partition property are closely related:

- $\kappa \xrightarrow{c.u.b.} (\kappa)^\lambda$ *implies* $\kappa \rightarrow (\kappa)^\lambda$.
- $\kappa \rightarrow (\kappa)^{\omega \cdot \lambda}$ *implies* $\kappa \xrightarrow{c.u.b.} (\kappa)^\lambda$.

We say κ has the strong partition property if $\kappa \rightarrow (\kappa)^\kappa$, and κ has the weak partition property if $\forall \lambda < \kappa \; \kappa \rightarrow (\kappa)^\lambda$.

The two variations of the partition property agree for κ having the weak or strong partition property.

Let $[\kappa]^\lambda_*$ denote the collection of functions from λ to κ of the correct type. We henceforth adopt the c.u.b. version of the partition relations.

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Assuming AD we have:

- The κ having the strong partition property are cofinal in Θ.
- ω_1 has the partition property, and more generally all the $\boldsymbol{\delta}^1_{2n+1}$ have the strong partition property.
- ω_2 has the weak partition propery, and more generally all the $\boldsymbol{\delta}^1_{2n+2}$ have the weak partition property.

If $\kappa \to (\kappa)^\lambda$ then we have a natural measure μ^λ_κ on $[\kappa]^\lambda_*$, the functions from λ to κ of the correct type.

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Fact

If $\kappa \to (\kappa)^\lambda$, then the measure μ_κ^λ is κ-complete.

If κ has the strong partition property and ν is a measue on κ, then there is a natural measure $S(\nu)$ on $j_\nu(\kappa)$ induced by the measure μ_κ^κ and the measure ν.

We can also define analogs $\tilde{S}(\nu)$ using functions of other uniform cofinalities.

For example, we have

- $S(W_1^1)$ is the ω-cofinal normal measure on ω_2
- $\tilde{S}(W_1^1)$ is the ω_1-cofinal normal measure on ω_2 (use f with $f(\alpha)$ of uniform cofinality α).

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A result of Martin says that for any measure ν, $j_\nu(\kappa)$ is a cardinal, and for any normal measure ν, $j_\nu(\kappa)$ is regular.

When ν is normal, the measure $S(\nu)$ is monotonic: if $f: j_\nu(\kappa) \to \mathrm{On}$, then there is a measure one set $A \subseteq j_\nu(\kappa)$ such that $f \restriction A$ is monotonically increasing.

The proof uses a partiton of pairs (f, g) where $f(\alpha) < g(\alpha) < f(\alpha + 1)$ and a "sliding argument."

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Theorem

Suppose $\kappa \to (\kappa)^\lambda$. *Then the measure* μ^λ_κ *is monotonic. That is if* $\Phi \colon [\kappa]^\lambda_* \to On$ *then there is a c.u.b.* $C \subseteq \kappa$ *such that if* $f, g \in [C]^\lambda_*$ *and* $f(\alpha) \le g(\alpha)$ *for all* $\alpha < \lambda$*, then* $\Phi(f) \le \Phi(g)$.

As a corollary we have:

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If κ has the strong partition property, and ν is a measure on κ, then the measure $S(\nu)$ on $j_\nu(\kappa)$ is monotonic.

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If κ has the strong partition property, and ν is a measure on κ, then the measure $S(\nu)$ on $j_\nu(\kappa)$ is monotonic.

We first prove the following special case of the theorem. We assume $\kappa \to (\kappa)^\kappa$.

Lemma

If $\Phi\colon [\kappa]^\kappa_* \to \mathrm{On}$ then there is a c.u.b. $C \subseteq \kappa$ such that if $f, g \in [C]^\kappa_*$ and for all $\alpha < \kappa$ we have:

1. $f(\alpha) \leq g(\alpha)$
2. $g(\alpha) \neq \sup_{\beta<\alpha} f(\beta)$ *for all limit* $\beta < \kappa$.
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We use functions of "indecomposable type." Fix $\mathcal{I} : \kappa \to \kappa$ increasing, discontinuous, with range in the indecomposable ordinals.

For $h \in [\kappa]^\kappa_*$, let $\text{main}(h)(\alpha) = h(\mathcal{I}(\alpha))$. Note that $\text{main}(h)$ is also of the correct type.

$\mathcal{P}$: partition $h \in [\kappa]^\kappa_*$ according to whether

$$\forall p \in [h[\kappa]]^\kappa_* \ \Phi(\text{main}(h)) \leq \Phi(\text{main}(p))$$

By wellfoundedness, on the homogeneous side of the partition the stated property holds. Fix C_0 homogeneous for $\mathcal{P}$ and let $C_1 \subseteq C_0$ be the closure points of C_0.

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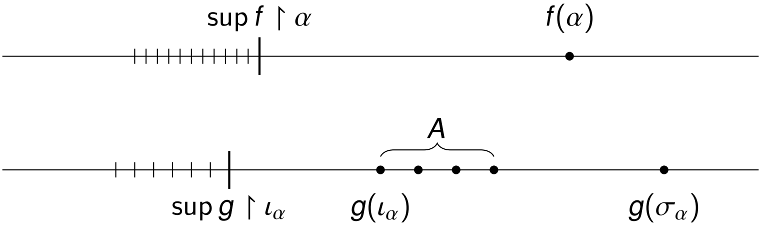
Fix $f, g \in [C_1]^\kappa_*$ satisfying (1)-(3), and we show that $\Phi(f) \leq \Phi(g)$.

We define two functions h, p with $h \in [C_0]^\kappa_*$, $p \in [h[\kappa]]^\kappa_*$, $\text{main}(h) = f$, and $\text{main}(p) = g$.

Let σ_β be least so that $g(\sigma_\beta) > f(\beta)$. So $\sigma_\beta \leq \beta + 1$.

Proceeding inductively on α we assume:

- For all $\beta < \alpha$, $h \restriction (\mathcal{I}(\beta) + 1)$ has been defined (of correct type) and $h(\mathcal{I}(\beta)) = f(\beta)$.
- For all $\beta < \alpha$, for all $\eta < \sigma_\beta$, $p \restriction (\mathcal{I}(\eta) + 1)$ has been defined and $p(\mathcal{I}(\eta)) = g(\eta)$.



- $\iota_\alpha = \sup_{\beta<\alpha} \sigma_\beta \leq \alpha$.
- $A = \{\beta \colon \sup f \restriction \alpha < g(\beta) < f(\alpha)\}$, o.t.$(A) = \xi$.
- If $A \neq \emptyset$ then $\iota_\alpha < \alpha$.
- $\sup f \restriction \alpha < g(\iota_\alpha)$ and $\sup(A) < f(\alpha)$ from our assumptions.

Let

$$\delta_0 = \sup\{\mathcal{I}(\beta) + 1 \colon \beta < \alpha\}$$
$$\tau_0 = \sup\{\mathcal{I}(\beta) + 1 \colon \beta < \iota_\alpha\}$$

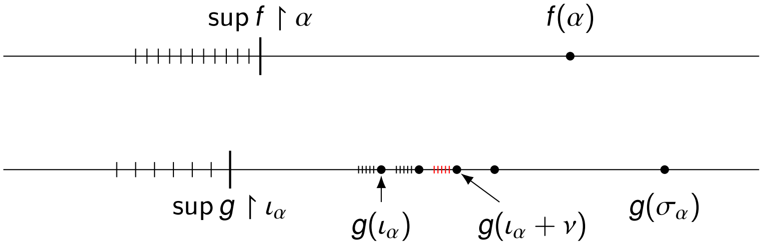
So we have defined $h \restriction \delta_0$, $p \restriction \tau_0$, and $\sup h \restriction \delta_0 = \sup f \restriction \alpha$, $\sup p \restriction \tau_0 = \sup g \restriction \iota_\alpha$.

For $\nu < \xi$, set:

$$\delta_\nu = \sup\{\delta_0 + \mathcal{I}(\iota_\alpha + \eta) + 1 \colon \eta < \nu\}$$
$$\epsilon_\nu = \delta_0 + \mathcal{I}(\iota_\alpha + \nu) = \delta_\nu + \mathcal{I}(\iota_\alpha + \nu)$$

$$\tau_\nu = \sup\{\tau_0 + \mathcal{I}(\iota_\alpha + \eta) + 1 \colon \eta < \nu\}$$
$$\mu_\nu = \tau_0 + \mathcal{I}(\iota_\alpha + \nu) = \tau_\nu + \mathcal{I}(\iota_\alpha + \nu)$$

Assume $h \restriction \delta_\nu$, $p \restriction \tau_\nu$ have been defined and $\sup h \restriction \delta_\nu = \sup p \restriction \tau_\nu = \sup g \restriction (\iota_\alpha + \nu)$.



For $\beta < \mathcal{I}(\iota_\alpha + \nu)$, define:

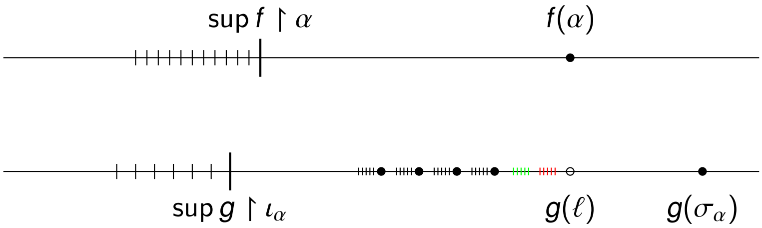
$$h(\delta_\nu + \beta) = p(\tau_\nu + \beta) = \mathbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \restriction \delta\nu)$$

This defines $h \restriction \epsilon_\nu$, $p \restriction \mu_\nu$, and we then set $h(\epsilon_\nu) = p(\mu_\nu) = g(\iota_\alpha + \nu)$.

Let $\delta = \sup\{\epsilon_\nu + 1 \colon \nu < \xi\}$, $\tau = \sup\{\mu_\nu + 1 \colon \nu < \xi\}$.

So, $h \restriction \delta$, $p \restriction \tau$ have been defined and $\sup f \restriction \delta = \sup g \restriction \tau = \sup g \restriction (\iota_\alpha + \xi)$.

- Note that $\tau \leq \delta \leq \delta_0 + \sup(\mathcal{I} \restriction \alpha) + 1 < \delta_0 + \mathcal{I}(\alpha) = \mathcal{I}(\alpha)$.



Let $\ell = \min(\kappa \setminus A) = \iota_\alpha + \xi$. We could have $g(\ell) > f(\alpha)$ or $g(\ell) = f(\alpha)$.

If $g(\ell) > f(\alpha)$, set for $\beta < \mathcal{I}(\alpha)$: $h(\delta + \beta) = \textbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \restriction \delta)$ and set $h(\mathcal{I}(\alpha)) = f(\alpha)$.

If $g(\ell) = f(\alpha)$ and $\ell = \alpha$, set for $\beta < \mathcal{I}(\alpha)$,

$h(\delta + \beta) = \textbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \restriction \delta)$,

$p(\tau + \beta) = \textbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup p \restriction \tau)$, If $g(\ell) = f(\alpha)$ and $\ell < \alpha$, set for $\beta < \mathcal{I}(\ell)$,

$$h(\delta + \beta) = \mathbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \restriction \delta),$$
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and for $\beta < \mathcal{I}(\alpha)$ set

$$h(\delta + \mathcal{I}(\ell) + \beta) = \mathbf{next}_{C_0}^{\omega\cdot(\beta+1)}(\sup h \restriction \mathcal{I}(\ell)),$$

as shown.

Set $h(\mathcal{I}(\alpha)) = f(\alpha)$, $p(\mathcal{I}(\ell)) = g(\ell) = f(\alpha)$.

General Case

We now consider the general case, without the restrictions on f and g.

Let C_0 be homogeneous for the previous restricted version, and C_1 the closure points of C_0.

Fix $f, g \in [C_1]^\kappa_*$ with $f(\alpha) \leq g(\alpha)$ for all $\alpha < \kappa$.

- We first lower g to get k with $f \leq k \leq g$ and such that (k, g) satisfies the assumptions and (f, k) satisfies $k(\alpha)$ is not of the form $\sup f \restriction \beta$ for limit β.
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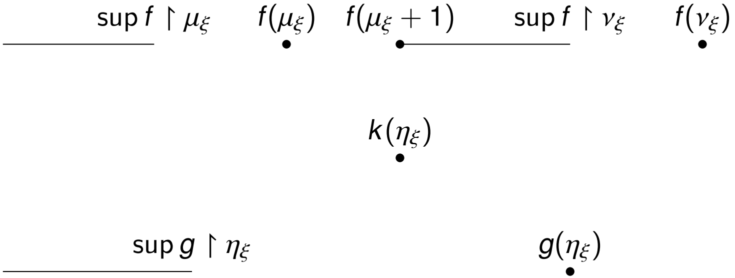
Definition of k:

Let (η_ξ, ν_ξ) enumerate the pairs with $g(\eta_\xi) = \sup f \restriction \nu_\xi$.

If α is not of the form η_ξ, let $k(\alpha) = g(\alpha)$.



We have $\eta_\xi \leq \mu_\xi < \mu_\xi + 1 < \nu_\xi$.

Definition of h:

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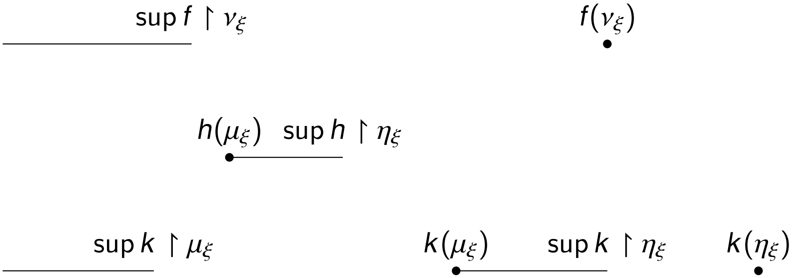

μ_ξ is least so that $k(\mu_\xi) > \sup f \restriction \nu_\xi$.
$\mu_\xi < \eta_\xi \leq \nu_\xi$.

Continuity

Theorem

Suppose $\epsilon < \kappa$, cof$(\epsilon) = \omega$, and $\kappa \to (\kappa)^{\epsilon \cdot \epsilon}$. Then for any $\Phi \colon [\kappa]^{\epsilon}_{*} \to On$, there is a c.u.b. $C \subseteq \kappa$ and a $\delta < \epsilon$ such that if $f, g \in [C]^{\epsilon}_{*}$ with $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

We have the following application.

Theorem

Suppose $\kappa \to (\kappa)^{<\kappa}$. Then for all $\lambda < \kappa$, there does not exist an injection of $\kappa^{<\kappa}$ into ${}^{\lambda}On$.

Theorem

Suppose $\epsilon < \kappa$, cof$(\epsilon) = \omega$, and $\kappa \to (\kappa)^{\epsilon\cdot\epsilon}$. Then for any $\Phi\colon [\kappa]^{\epsilon}_{*} \to On$, there is a c.u.b. $C \subseteq \kappa$ and a $\delta < \epsilon$ such that if $f, g \in [C]^{\epsilon}_{*}$ with $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

We have the following application.

Theorem

Suppose $\kappa \to (\kappa)^{<\kappa}$. Then for all $\lambda < \kappa$, there does not exist an injection of $\kappa^{<\kappa}$ into ${}^{\lambda}On$.

Continuity

Theorem

Suppose $\epsilon < \kappa$, cof$(\epsilon) = \omega$, and $\kappa \to (\kappa)^{\epsilon \cdot \epsilon}$. Then for any $\Phi \colon [\kappa]^{\epsilon}_{*} \to On$, there is a c.u.b. $C \subseteq \kappa$ and a $\delta < \epsilon$ such that if $f, g \in [C]^{\epsilon}_{*}$ with $f \restriction \delta = g \restriction \delta$ and $\sup(f) = \sup(g)$, then $\Phi(f) = \Phi(g)$.

We have the following application.

Theorem

Suppose $\kappa \to (\kappa)^{<\kappa}$. Then for all $\lambda < \kappa$, there does not exist an injection of $\kappa^{<\kappa}$ into ${}^{\lambda}On$.

We prove the application from the theorem.

Proof: Suppose $\Phi\colon \kappa^{<\kappa} \to {}^{\lambda}\mathrm{On}$ is injective.

For each $\gamma < \lambda$ and ϵ, κ, Φ induces $\Phi^{\epsilon}_{\gamma}\colon [\kappa]^{\epsilon}_{*} \to \mathrm{On}$ by

$$\Phi^{\epsilon}_{\gamma}(f) = \Phi(f)(\gamma)$$

By the Theorem, $\forall \gamma < \lambda\ \forall \epsilon < \kappa\ \exists C \subseteq \kappa\ \exists \delta < \epsilon$ for all $f, g \in [C]^{\epsilon}_{*}$, if $\sup f = \sup g$ and $f \restriction \delta = g \restriction \delta$, then $\Phi^{\epsilon}_{\gamma}(f) = \Phi^{\epsilon}_{\gamma}(g)$.

Let $\delta^{\epsilon}_{\gamma} < \epsilon$ be the least such δ.

For each $\gamma < \lambda$, let $\delta_\gamma < \kappa$ be such that for almost all ϵ of cofinality ω, we have $\delta_\gamma^\epsilon = \delta_\gamma$.

Let $\delta^* = \sup_{\gamma<\lambda} \delta_\gamma < \kappa$.

For each $\gamma < \lambda$, there is an ω-club in κ of ϵ such that $\delta_\gamma^\epsilon = \delta_\gamma < \delta^*$.

By additivity of the club filter, we may fix an $\epsilon^* < \kappa$ so that for all $\gamma < \lambda$, $\delta_\gamma^{\epsilon^*} < \delta^*$.

So, for all $\gamma < \lambda$ there is a club $C \subseteq \kappa$ such that for all $f, g \in [C]_*^{\epsilon^*}$ with $\sup f = \sup g$ and $f \restriction \delta^* = g \restriction \delta^*$ we have $\Phi(f)(\gamma) = \Phi(g)(\gamma)$.

We need a variation of the additivity argument.

If we find a c.u.b. $C \subseteq \kappa$ that works for all $\gamma < \lambda$, then we have a contradiction:

Consider $f, g \in [C]^{\epsilon^*}_*$ with $\sup f = \sup g$, $f \upharpoonright \delta^* = g \upharpoonright \delta^*$ and with $f \neq g$.

Additivity argument.

For all $\gamma < \lambda$, $\forall^* f \in [\kappa]^{\epsilon^*}_*$ if $g \upharpoonright \delta^* = f \upharpoonright \delta^*$, and $g \sqsubseteq f$, then $\Phi(g)(\gamma) = \Phi(g)(\gamma)$.

By the additivity of the function space measure, there is a c.u.b. $C \subseteq \kappa$ such that $\forall \gamma < \lambda\ \forall f \in [C]^{\epsilon^*}_*$ if $g \upharpoonright \delta^* = f \upharpoonright \delta^*$, and $g \sqsubseteq f$, then $\Phi(f)(\gamma) = \Phi(g)(\gamma)$.

This c.u.b. $C \subseteq \kappa$ works. If $f, g \in [C]^{\epsilon^*}_*$, $\sup f \upharpoonright \delta^* = \sup g \upharpoonright \delta^*$, and $\sup(f) = \sup(g)$, then there is an $h \in [C]^{\epsilon^*}_*$ with $h \upharpoonright \delta^* = f, g \upharpoonright \delta^*$, and with $f \sqsubseteq h$, $g \sqsubseteq h$.