Introduction to choiceless large cardinals

Gabriel Goldberg

UC Berkeley

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Outline

- Limits of the large cardinal hierarchy in ZFC
- Large cardinals beyond choice
- Structure theory of embeddings from V to V

Large cardinals

Natural hierarchy of theories extending and strengthening the standard ZF(C) axioms: large cardinal hypotheses

- Large cardinals provide strong background theories necessary to analyze set theoretic principles
- Taken as axioms, large cardinals provide a rich and coherent picture of the universe of sets

Kunen's theorem

Taken to its natural extreme, the modern paradigm for formulating large cardinal hypotheses produces principles that are inconsistent with the Axiom of Choice

- The consistency of hypotheses at this level cannot be proved using large cardinal hypotheses compatible with AC
- Taken as axioms, these choiceless large cardinal hypotheses are starting to provide a rich and coherent picture... of what?

Elementary embeddings and large cardinals

Theorem (Scott)

A cardinal κ is measurable if and only if one can find an inner model $M \subseteq V$ and an elementary embedding $j : V \to M$ such that κ is the critical point of j.

- For such a $j : V \to M$, one has $V_{\kappa+1} \subseteq M$ and $j \restriction V_\kappa = \text{id}$
- κ is 2-**strong** if one can in addition obtain $V_{\kappa+2} \subseteq M$
- κ is **superstrong** if $j(V_\kappa) = V_{j(\kappa)}$
  - In general, one only knows $j(V_\kappa) = V_{j(\kappa)} \cap M$
- κ is n-**fold superstrong** if $j^n(V_\kappa) = V_{j^n(\kappa)}$

Theorem (Kunen)

There is no nontrivial elementary embedding from V to V.

Kunen's bound

Suppose P, Q are transitive and $j : P \to Q$ is elementary

- For $n < \omega$, $\kappa_n(j) = j^n(\mathrm{crit}(j))$
- $\kappa_\omega(j) = \sup_{n<\omega} \kappa_n(j)$

So κ is n-fold superstrong iff there is $j : V \to M$ with $\mathrm{crit}(j) = \kappa$ and $V_{\kappa_n(j)} \subseteq M$

Axiom $\mathbf{I}_2$: κ is ω-**fold superstrong** if there is a $j : V \to M$ with critical point κ such that $V_{\kappa_\omega(j)} \subseteq M$

Theorem (Kunen)

There is no elementary embedding $j : V \to M$ such that $V_{\kappa_\omega(j)+1} \subseteq M$.

Rank-into-rank embeddings

Another way to climb the large cardinal hierarchy:
embeddings from levels of the cumulative hierarchy to itself

- If $j : V \to M$ is elementary with $V_\lambda \subseteq M$ where $\lambda = \kappa_\omega(j)$, $j \restriction V_\lambda$ is an elementary embedding from V_λ to itself
- Despite Kunen's bound that $V_{\lambda+1} \not\subseteq M$, elementary $i : V_{\lambda+1} \to V_{\lambda+1}$ (Axiom I_1) are *not* believed inconsistent with ZFC
- On the other hand, nontrivial elementary embeddings from $V_{\lambda+2}$ to itself *are* inconsistent with ZFC

This has led researchers to focus on the "edge" of inconsistency, just past $V_{\lambda+1}$.

The edge of inconsistency

The principle $\mathbf{D}_n(\lambda)$ states that there is a Σ_n-elementary embedding from $V_{\lambda+1}$ to $V_{\lambda+1}$ with $\lambda = \kappa_\omega(j)$.

Theorem (Martin)

$\mathrm{D}_0(\lambda) < \mathrm{D}_1(\lambda) = \mathrm{D}_2(\lambda) < \mathrm{D}_3(\lambda) = \mathrm{D}_4(\lambda) < \cdots$

$\mathrm{D}_0(\lambda)$ is equivalent to an elementary $j : V_\lambda \to V_\lambda$, and $\mathrm{D}_1(\lambda)$ is equivalent to ω-superstrength.

The edge of inconsistency

Axiom $\mathbf{D}^1_n(\lambda)$: Σ_n-elementary embedding from $(V_{\lambda+1}, T)$ to itself, where T denotes the satisfaction predicate of $V_{\lambda+1}$.

Axiom $\mathbf{D}^2_n(\lambda)$: Σ_n-elementary embedding from $(V_{\lambda+1}, S)$ to itself, where S denotes the satisfaction predicate of $(V_{\lambda+1}, T)$.

Axiom $\mathbf{D}^\alpha_n(\lambda)$: Σ_n-elementary embedding from $L_\alpha(V_{\lambda+1})$ to itself.

Woodin's axiom $\mathbf{I}_0(\lambda)$: there is an elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\lambda = \kappa_\omega(j)$.

$L(\mathbb{R})$ and $L(V_{\lambda+1})$

Theorem (Solovay)

Under $\mathrm{AD}^{L(\mathbb{R})}$, ω_1 *is measurable in* $L(\mathbb{R})$.

Theorem (Woodin)

Under $\mathrm{I}_0(\lambda)$, λ^+ *is measurable in* $L(V_{\lambda+1})$.

$L(\mathbb{R})$ and $L(V_{\lambda+1})$

Theorem (Davis)

In $L(\mathbb{R})$ under $\mathsf{AD}^{L(\mathbb{R})}$, every uncountable subset of $\mathbb{R}$ is in bijection with $\mathbb{R}$.

Theorem (Cramer)

In $L(V_{\lambda+1})$ under $\mathsf{I}_0(\lambda)$, every subset of $V_{\lambda+1}$ of cardinality greater than λ is in bijection with $V_{\lambda+1}$.

$L(\mathbb{R})$ and $L(V_{\lambda+1})$

Let $\Theta^{L(\mathbb{R})}$ denote the supremum of all ordinals α that are the surjective image of $\mathbb{R}$ in $L(\mathbb{R})$.

Theorem (Moschovakis)

Under $\mathrm{AD}^{L(\mathbb{R})}$, $\Theta^{L(\mathbb{R})}$ is a strongly inaccessible cardinal in $L(\mathbb{R})$: i.e., for all $\eta < \Theta$, every function from $P(\eta)$ to Θ is bounded.

Let $\Theta^{L(V_{\lambda+1})}$ denote the supremum of all ordinals α that are the surjective image of $V_{\lambda+1}$ in $L(V_{\lambda+1})$.

Theorem (Woodin)

Under $\mathrm{I}_0(\lambda)$, $\Theta^{L(V_{\lambda+1})}$ is a strongly inaccessible cardinal in $L(V_{\lambda+1})$.

$L(\mathbb{R})$ and $L(V_{\lambda+1})$

Theorem (Kunen)

Under $\mathrm{AD}^{L(\mathbb{R})}$, *in* $L(\mathbb{R})$, *every* ω_1*-complete filter on an ordinal less than* Θ *extends to an* ω_1*-complete ultrafilter.*

Theorem (G.)

Under $\mathrm{I}_0(\lambda)$, *in* $L(V_{\lambda+1})$, *every* λ^+*-complete filter on an ordinal less than* Θ *extends to a* λ^+*-complete ultrafilter.*

Beyond $L(V_{\lambda+1})$

Theorem (Cramer)

If there is an elementary embedding from $L(V^{\#}_{\lambda+1})$ to itself, then there is an ω-club of $\gamma < \lambda$ such that $\mathsf{I}_0(\gamma)$ holds.

Analogy between $L(\mathbb{R})$ and $L(V_{\lambda+1})$ extends:

- Determinacy in $L(\mathbb{R}, \Gamma)$ for pointclasses $\Gamma \subseteq P(\mathbb{R})$
- Elementary embeddings of $L(V_{\lambda+1}, \Gamma)$ for $\Gamma \subseteq P(V_{\lambda+1})$

E.g., Woodin constructed the $L(V_{\lambda+1}, \Gamma)$ analog of the minimum model of $\mathrm{AD}_{\mathbb{R}}$; i.e., $L(\mathbb{R}, \Gamma)$ where $\Gamma \subseteq P(\mathbb{R})$ is Wadge minimal such that every game on $\mathbb{R}$ in Γ is determined via a Γ-strategy.

Beyond AC

Choiceless large cardinal hypotheses appear to be stronger than all of these principles. For example:

Theorem (G.)

ZF + DC + $j : V_{\lambda+3} \to V_{\lambda+3}$ *proves* Con(ZFC + I_0).

This is just the base of what seems to be an endless hierarchy

- λ is **rank Berkeley** if for all $\alpha < \lambda \leq \beta$, there is an elementary embedding $j : V_\beta \to V_\beta$ with $\alpha < \text{crit}(j) < \lambda$
- δ is **Berkeley** if for all transitive $M \supseteq \delta$, there is an elementary $j : M \to M$ with $\alpha < \text{crit}(j) < \delta$

If $j : V \to V$ is elementary, $\kappa_\omega(j)$ is rank Berkeley. In ZF alone, a Berkeley cardinal proves the consistency of ZFC + I_0 and probably of all ZFC large cardinals ever studied (due to Woodin).

Consistency of choiceless large cardinals

How can one get a handle on the consistency of choiceless large cardinals?

- If the choiceless hierarchy really outstrips the ZFC large cardinal hierarchy, the usual methodology is not available
- One can attempt to refute choiceless cardinals in ZF
- Or one can attempt to develop their "structure theory"

The HOD conjecture

Theorem (Jensen)

Exactly one of the following holds:

- *Every uncountable set of ordinals is contained in a constructible set of the same cardinality.*
- *Every uncountable cardinal is strongly inaccessible in L.*

Theorem (Woodin)

If κ is extendible, exactly one of the following holds:

- *Every set of ordinals of size at least κ is contained in an ordinal definable set of the same cardinality.*
- *Every regular cardinal above κ is measurable in* HOD.

Woodin's HOD conjecture (assuming large cardinals): there is a proper class of regular cardinals that are not measurable in HOD.

Unique embeddings

For P, Q transitive and $\delta \in \mathrm{Ord}^P$, the embeddings $j_0, j_1 : P \to Q$ are δ-**similar** if $j_0(\delta) = j_1(\delta)$ and $\sup_{\alpha<\delta} j_0(\alpha) = \sup_{\alpha<\delta} j_1(\alpha)$.

Theorem

If κ is supercompact, then the following are equivalent:

- *For all regular $\delta \geq \kappa$, for some $\alpha > \delta$, if $j_0, j_1 : V_\alpha \to M$ are δ-similar elementary embeddings, then $j_0 \restriction \delta = j_1 \restriction \delta$.*
- *The* HOD *conjecture is true.*

Theorem

If M is an inner model and $j_0, j_1 : V \to M$ are elementary embeddings, then $j_0 \restriction \mathrm{Ord} = j_1 \restriction \mathrm{Ord}$.

Schlutzenberg's theorem

Theorem (Schlutzenberg)

ZFC + $\mathsf{I}_0(\lambda)$ *is equiconsistent with* ZF + DC_λ + *the existence of an elementary embedding from* $V_{\lambda+2}$ *to* $V_{\lambda+2}$.

Major open question: What about $V_{\lambda+3}$? Or $j : V \to V$?

Theorem (Schlutzenberg)

Suppose $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ *is elementary with* $\lambda = \kappa_\omega(j)$. *Let* $i = j \restriction (V_{\lambda+2})^{L(V_{\lambda+1})}$ *and let* $M = L(V_{\lambda+1}, i)$. *Then*

$$(V_{\lambda+2})^M = (V_{\lambda+2})^{L(V_{\lambda+1})}$$

Therefore in M, i *witnesses that there is an elementary embedding from* $V_{\lambda+2}$ *to itself.*

Periodicity in the cumulative hierarchy

Suppose λ is a limit ordinal.

- If $j : V_{\lambda+1} \to V_{\lambda+1}$ is elementary, j is definable over $V_{\lambda+1}$ from the parameter $i = j \restriction V_\lambda$: for any $X \in V_{\lambda+1}$,

$$j(X) = \bigcup_{\alpha<\lambda} i(X \cap V_\alpha)$$

- If $j : V_{\lambda+2} \to V_{\lambda+2}$, then j is not boldface definable over $V_{\lambda+2}$
- Schlutzenberg asked: for $n \geq 3$, must $j : V_{\lambda+n} \to V_{\lambda+n}$ be undefinable over $V_{\lambda+n}$?

Theorem (G., Schlutzenberg)

Suppose $j : V_\alpha \to V_\alpha$ are elementary embeddings. Then j is definable over V_α if and only if α is an odd ordinal.

Reflection and collection

A cardinal κ is Σ_2-**reflecting** if $V_\kappa \preceq_{\Sigma_2} V$.

Theorem (Wellordered collection lemma)

Suppose λ is rank Berkeley and $\kappa \geq \lambda$ is a singular Σ_2-reflecting cardinal. If $\mathcal{F}$ is a family of nonempty sets with $|\mathcal{F}| \leq \kappa$, then there is a set $\{a_x : x \in V_\kappa\}$ that intersects every set in $\mathcal{F}$.

Note that the Axiom of Choice for families of size λ^+ is false if there is an elementary embedding from $V_{\lambda+2}$ to itself.

Corollary

If λ is rank Berkeley and $\kappa \geq \lambda$ is Σ_2-reflecting, either κ or κ^+ is regular.

Measures on ordinals

Theorem

Suppose λ is rank Berkeley and $\kappa \geq \lambda$ is Σ_2-reflecting. Then every set of κ-complete ultrafilters on ordinals can be wellordered.

This is analogous to the AD theorem that the set of ultrafilters on ordinals less than $\theta_{\omega+2}$ can be wellordered.

Theorem

Suppose λ is rank Berkeley and $\kappa \geq \lambda$ is Σ_2-reflecting. Then every κ-complete filter on an ordinal extends to a κ-complete ultrafilter.

Club filters

A filter F is **atomic** if every F-positive set S has an F-positive subset T such that $F \cup \{T\}$ generates an ultrafilter.

Theorem

If λ is rank Berkeley and $\kappa \geq \lambda$ is Σ_2-reflecting, then the club filter on any regular cardinal above κ is κ-complete and atomic.

The structure of the club filter in this context is very similar to the expected structure under AD.

Corollary

If λ is rank Berkeley and $\kappa \geq \lambda$ is Σ_2-reflecting, either κ or κ^+ is measurable.

Lindenbaum numbers

If α is an ordinal, then θ_α denotes the supremum of all ordinals η that are the surjective image of a set in V_α.

- $\theta_\omega = \omega$
- $\theta_{\omega+1} = \omega_1$
- $\theta_{\omega+2} = \mathfrak{c}^+$ assuming AC
  - More generally, $\theta_{\omega+\alpha+1} = (\beth_\alpha)^+$ assuming AC
  - So under AC, $\theta_{\omega+\alpha} = \aleph_\alpha$ for all ordinals α iff GCH holds
- $(\theta_{\omega+2})^{L(\mathbb{R})} = \Theta^{L(\mathbb{R})}$
- If α is a limit ordinal, θ_α is a strong limit and $\theta_{\alpha+1} = (\theta_\alpha)^+$
- Under AD, $\theta_{\omega+2}$ is a strong limit and $\theta_{\omega+3} = (\theta_{\omega+2})^+$

Periodicity, continued

If α is an ordinal, then θ_α denotes the supremum of all ordinals η that are the surjective image of a set in V_α.

Theorem

If λ is rank Berkeley and $\kappa \geq \lambda$ is Σ_2-reflecting, then for all even ordinals $\epsilon \geq \kappa$:

- *θ_ϵ is a strong limit cardinal: if $\eta < \theta_\epsilon$, then θ_ϵ is not the surjective image of $P(\eta)$.*
- *$\theta_{\epsilon+1}$ is not a strong limit cardinal: in fact, $\theta_{\epsilon+1}$ is the surjective image of $P(\theta_\epsilon)$.*

Open question: is $\theta_{\epsilon+1} = (\theta_\epsilon)^+$?

- One can show $|\text{Reg} \cap (\theta_\epsilon, \theta_{\epsilon+1})| < \lambda$

Plan

I'll outline a proof of the periodicity theorem for the θ_α-sequence. This will require:

- Periodicity for definability of embeddings
- Counting ultrafilters on ordinals
  - Wellordered collection lemma
  - Hartogs numbers: $\aleph(V_{\epsilon+2}) = \theta_{\epsilon+2}$
  - Analogs of the Moschovakis coding lemma for $\text{HOD}_{V_{\epsilon+1}}$
- Some choiceless theory of ultrapowers and extenders

Thanks