The negation of the weak Kurepa hypothesis and guessing models

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Guessing models

Definition

Let $\theta \geq \omega_2$ be a regular cardinal and let $M \prec H(\theta)$ have size ω_1.

1. Given a set $x \in M$, and a subset $d \subseteq x$, we say that
   1. d is *(μ, M)-approximated* for $\omega < \mu \leq \omega_2$ if, for every $z \in M \cap \mathscr{P}_\mu(M)$, we have $d \cap z \in M$;
   2. d is *M-guessed* if there is $e \in M$ such that $d \cap M = e \cap M$.
2. M is a *μ-guessing model for x* if every (μ, M)-approximated subset of x is M-guessed.
3. M is a *μ-guessing model* if M is μ-guessing for every $x \in M$.

Guessing model property

Definition

We denote by $\mathrm{GMP}(\mu, \omega_2, H(\theta))$ the assertion that the set

$$\{M \in \mathscr{P}_{\omega_2}(H(\theta)) \,|\, M \text{ is a } \mu\text{-guessing model}\}$$

is stationary in $\mathscr{P}_{\omega_2}(H(\theta))$.

Kurepa and weak Kurepa trees

Recall the following definition:

Definition

Let κ be a cardinal. We say that a κ-tree T is a κ-*Kurepa tree* if it has at least κ^+-many cofinal branches; if we drop the restriction on T being a κ-tree, and require only that T has size and height κ, we obtain a *weak Kurepa tree*. We say that the *Kurepa Hypothesis*, KH(κ), holds if there exists a Kurepa tree on κ; analogously the *weak Kurepa Hypothesis*, wKH(κ), says that there exists a weak Kurepa tree on κ.

GMP and Kurepa trees

Cox and Krueger showed that $\mathrm{GMP}(\omega_1, \omega_2, H(\omega_2))$ implies $\neg\mathrm{wKH}(\omega_1)$. We show that this is not the case for $\mathrm{GMP}(\omega_2, \omega_2, H(\omega_2))$.

Theorem (Lambie-Hanson, S., 2022)

Let κ be a supercompact cardinal. Then there is a generic extension where the following hold:

1. $2^\omega = \kappa = \omega_2$;
2. *there is an ω_1-Kurepa tree;*
3. $\mathrm{GMP}(\omega_2, \omega_2, H(\theta))$ *for every $\theta \geq \omega_2$.*

GMP and Kurepa trees

- In the proof we use a characterization of GMP formulated in terms of slender lists. Recall that $\mathrm{GMP}(\omega_2, \omega_2, H(\theta))$ holds for every $\theta \geq \omega_2$ if and only if $\mathrm{ISP}(\omega_2, \omega_2, \lambda)$ holds for every $\lambda \geq \omega_2$.
- Our proof is based on the proof of Cummings who showed that the tree property at ω_2 is consistent with the existence of ω_1-Kurepa trees.
- The model is obtained by forcing with the product of Mitchell forcing $\mathbb{M}(\omega, \kappa)$ up to a supercompact cardinal κ with the forcing for adding an ω_1-Kurepa tree with κ-many cofinal branches $\mathbb{K}(\omega_1, \kappa)$: $\mathbb{M}(\omega, \kappa) \times \mathbb{K}(\omega_1, \kappa)$.
- In the proof it is crucial that ω_2-Knaster forcings cannot add cofinal branches to ω_2-slender (ω_2, λ)-lists. This cannot be extended to ω_1-slender (ω_2, λ)-lists, by Cox and Krueger's theorem we mentioned above.

ω_2-guessing models that are not ω_1-guessing

- Viale asked whether it is consistent that there exist μ-guessing models that are not ω_1-guessing for some $\mu > \omega_1$.
- A positive answer was given by Hachtman and Sinapova; they proved that if μ is a singular limit of ω-many supercompact cardinals, then for every sufficiently large regular cardinal θ, there are stationarily many μ^+-guessing models in $\mathscr{P}_{\mu^+} H(\theta)$ that are not δ-guessing for any $\delta \leq \mu$.
- Our theorem provides a different way of showing this, and at a smaller cardinal and from a weaker large cardinal assumption.

ω_2-guessing models that are not ω_1-guessing

Corollary (Lambie-Hanson, S., 2022)

Let κ be a supercompact cardinal. There is a generic extension in which for all regular $\theta \geq \omega_2$, there are stationarily many ω_2-guessing models $M \in \mathscr{P}_{\omega_2} H(\theta)$ that are not ω_1-guessing.

weak Kurepa trees

Recall the definition of a weak Kurepa tree:

Definition

Let κ be a cardinal. We say that a tree T of size and height κ is a κ-*weak Kurepa tree* if it has at least κ^+-many cofinal branches. We say that the *weak Kurepa Hypothesis*, wKH(κ), holds if there exists a weak Kurepa tree on κ.

The negation of the weak Kurepa hypothesis

Some basic properties:

- If CH holds, then $2^{<\omega_1}$ is a weak Kurepa tree.
- Therefore $\neg$wKH(ω_1) implies $2^\omega > \omega_1$.
- (Mitchell) In the generic extension by Mitchell forcing up to an inaccessible cardinal $\neg$wKH(ω_1) holds.
- (Silver) The inaccessible cardinal is necessary. If $\neg$wKH(ω_1) holds, then ω_2 is inaccessible in L.

The negation of the weak Kurepa hypothesis

Assume that $\neg\mathsf{wKH}(\omega_1)$ holds:

- (Baumgartner) If $2^\omega = \omega_2$, then $2^{\omega_1} = \omega_2$; in fact, even $\Diamond^+(\omega_2 \cap \mathrm{cof}(\omega_1))$ holds.
- Baumgartner's result can be generalized as follows: if $2^\omega < \aleph_{\omega_1}$, then $2^{\omega_1} = 2^\omega$.
- This result is sharp, as we showed that if the existence of a supercompact cardinal is consistent, then it is consistent that $\neg\mathsf{wKH}(\omega_1)$ holds, $2^\omega = \aleph_{\omega_1}$, and $2^{\omega_1} > \aleph_{\omega_1+1}$.

Special trees

- Recall that a tree T of height ω_1 is *special* if there is a function $f : T \to \omega$ such that, for all $s, t \in T$, if $s <_T t$, then $f(s) \neq f(t)$. It is immediate that a special tree cannot have an uncountable branch.
- Baumgartner introduced a generalization of this notion of specialness that can also be satisfied by trees of height ω_1 that have uncountable branches; this notion was used to prove that PFA implies $\neg$wKH(ω_1).

B-special trees

Definition

Suppose that T is a tree of height ω_1. We say that T is *B-special* if there is a function $f : T \to \omega$ such that, for all $s, t, u \in T$, if $f(s) = f(t) = f(u)$ and $s <_T t, u$, then t and u are $<_T$-comparable.

- This is indeed a generalization of the notion of specialness, since if T is a tree of height ω_1 with no uncountable branches, then T is special if and only if T is B-special.

A strengthening of $\neg$wKH(ω_1) using B-special trees

- (Baumgartner) If T is a tree of size and height ω_1 which is B-special then T has at most ω_1 many cofinal branches. Therefore if all trees of size and height ω_1 are B-special, then $\neg$wKH(ω_1) holds.
- (Baumgartner) If T is tree of size and height ω_1 with at most ω_1 many cofinal branches, then there is a ccc forcing which forces that T is B-special.

A strengthening of $\neg$wKH(ω_1) using B-special trees

- (Baumgarter) $\mathsf{MA}_{\omega_1} + \neg\mathsf{wKH}(\omega_1)$ implies that every tree T of size and height ω_1 without cofinal branches is B-special.
- $\neg\mathsf{wKH}(\omega_1)+$ all trees of size and height ω_1 without cofinal branches are special $\Leftrightarrow$ all trees of size and height ω_1 are B-special.
- Baumgartner and Todorcevic independently proved that $\mathsf{MA}_{\omega_1} + \neg\mathsf{wKH}(\omega_1)$ is consistent relative to the existence of an inaccessible cardinal.

$\neg$wKH(ω_1) and 2^ω

- Note that $\neg$wKH(ω_1) is consistent with an arbitrary value of $2^\omega = \mu$. Consider the product of Mitchell forcing up to an inaccessible cardinal with Cohen forcing at ω of length μ.
- It is not obvious how to get all trees B-special on ω_1 with $2^\omega > \omega_2$. We cannot just use Cohen forcing over Todorcevic's or Baumgartner's model because Cohen forcing adds an ω_1-Suslin tree.

$\neg$wKH(ω_1) and 2^ω

- One could try Todorcevic's approach. He first forces with Mitchell forcing up to an inaccessible cardinal κ and then over the Mitchell model forces with the iteration (finite support) of length κ of small ccc forcings which do not add cofinal branches to trees of height and size ω_1, which ensures MA_{ω_1}.
- We do not know how to generalize this approach for finite support iterations of length $\mu > \omega_2 = \kappa$ for an arbitrary μ.
  - It works if the cofinality of μ is at least ω_2: then it is possible to specialize all trees without cofinal branches of height and size ω_1.
  - However, this does not solve the problem for $2^\omega = \mu$, where $\mathrm{cf}(\mu) = \omega_1$, because a tree may appear at the end of the iteration, without being added at some intermediate stage.

All trees B-special and possible values of 2^ω

- Laver found a way around this problem: he showed that if one forces with a measure algebra $\mathbb{B}$ over any model of MA_{ω_1}, then in the resulting forcing extension, it remains true that every tree of height and size ω_1 with no uncountable branches is special.
- We modified Laver's argument to prove an analogous result and showed that if one forces with a measure algebra over any model of PFA, then in the resulting forcing extension every tree of height and size ω_1 is B-special.

All trees B-special and possible values of 2^ω

Theorem (Lambie-Hanson, S., 2022)

Suppose that PFA *holds,* κ *is an infinite cardinal, and* $\mathbb{B}$ *is the measure algebra on* 2^κ*, with associated measure* μ*. Then, in* $V^{\mathbb{B}}$*, every tree of height and size* ω_1 *is* B*-special.*

- Recall the proof that PFA implies $\neg$wKH(ω_1).
- Let T be a tree of size and height ω_1 with ω_2-many cofinal branches.
- Consider the forcing $\mathrm{Coll}(\omega_1, \omega_2) * \mathrm{Spec(T)}$.
- Our proof uses this approach except we specialize not a tree, but a $\mathbb{B}$-name for a tree using Laver's method.

An "indestructible" version of $\neg$wKH(ω_1)

As a corollary, we can show that an "indestructible" version of the negation of the weak Kurepa Hypothesis is compatible with any possible value of the continuum except ω_1. More precisely:

Corollary (Lambie-Hanson, S., 2022)

Suppose that PFA *holds,* $\kappa \geq \omega_2$ *is a cardinal of uncountable cofinality, and* $\mathbb{B}$ *is the measure algebra on* 2^κ. *Then, in* $V^{\mathbb{B}}$, $2^\omega = \kappa$ *and, for every tree* T *of size and height* ω_1 *and every outer model* W *of* $V^{\mathbb{B}}$ *such that* $(\omega_1)^W = (\omega_1)^{V^{\mathbb{B}}}$, T *has at most* ω_1*-many uncountable branches in* W.

Indestructible guessing model property

Cox and Krueger introduce the *indestructible guessing model property*.

Definition

Let $\theta \geq \omega_2$ be a cardinal. Then IGMP(θ) is the assertion that there are stationarily many $M \in \mathscr{P}_{\omega_2}(H(\theta))$ such that M is an ω_1-guessing model and remains an ω_1-guessing model in any forcing extension that preserves ω_1. IGMP is the statement that IGMP(θ) holds for all $\theta \geq \omega_2$.

- By an argument of Cox and Krueger, who showed that IGMP implies that no tree of size and height ω_1 gets a new cofinal branch in any ω_1-preserving forcing extension, it is clear that IGMP(ω_2) implies the indestructible version of $\neg$wKH(ω_1) in the case in which W is a forcing extension of V.
- We saw that this indestructible version of $\neg$wKH(ω_1) is compatible with any possible value of the continuum, including values of cofinality ω_1.
- Cox and Krueger showed that IGMP is compatible with any possible value of the continuum with cofinality at least ω_2. The combination of these two results naturally raises question (1):

Open questions

Some open questions:

1. Is IGMP compatible with $\mathrm{cf}(2^\omega) = \omega_1$? What about just $\mathrm{IGMP}(\omega_2)$?
2. Does IGMP imply that all trees of size and height ω_1 are B-special? Or even, does the "indestructible" version of $\neg\mathrm{wKH}(\omega_1)$ imply that all trees of size and height ω_1 are B-special?