A tail of a generic real
Classifying invariants for E_1

Assaf Shani

Harvard University

Advances in Set Theory
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We will see some structural results about this model.
The main topic of this talk is: what do the properties of this model tell us about E_1?

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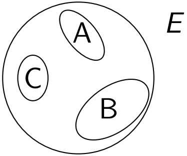

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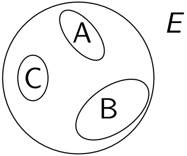

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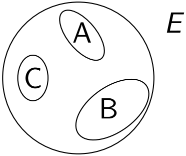

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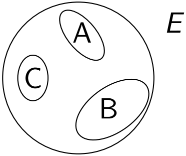

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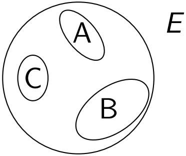

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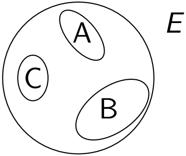

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E,F E.R.s on Polish spaces X, Y. $f\colon X \to Y$ is a **reduction** if

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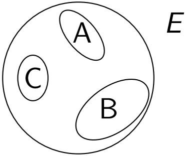

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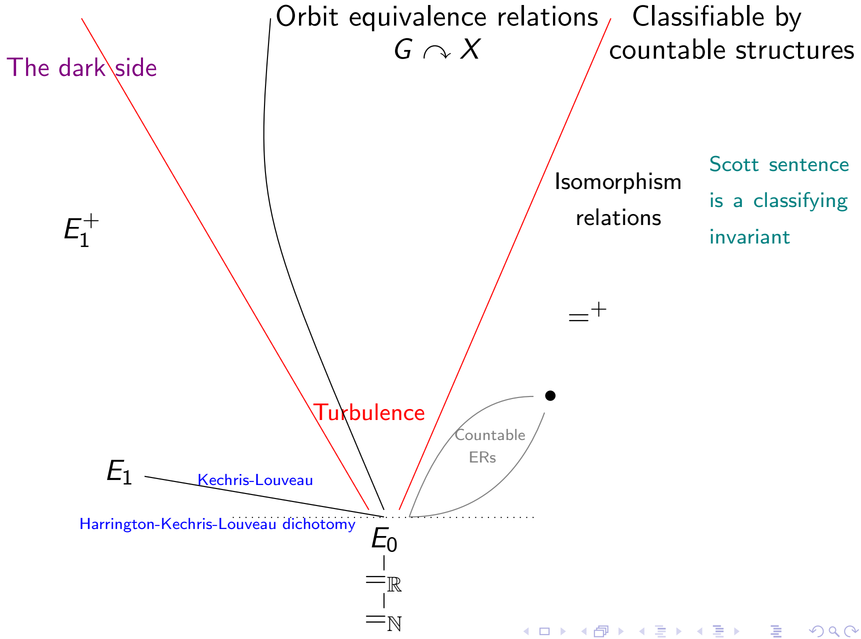
$\implies$ *Classifying invariants for F can be used to classify E.*

An extremely partial picture of Borel equivalence relations



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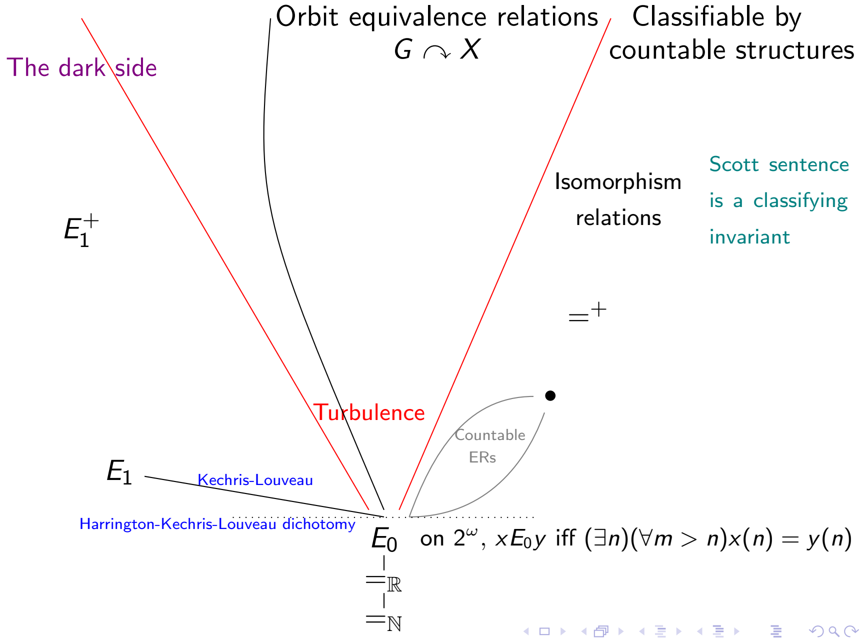

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Question: is (1) optimal? Cichon-Pawlikowsky: $\mathfrak{b}^{V[x]} = \mathbf{add}(\mathcal{B})^V$

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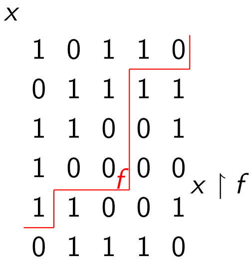

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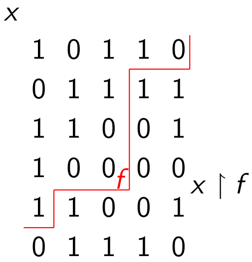

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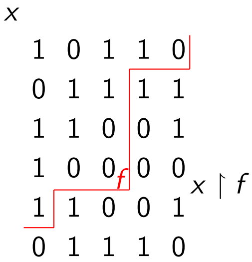

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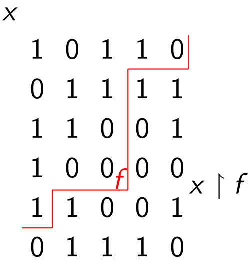

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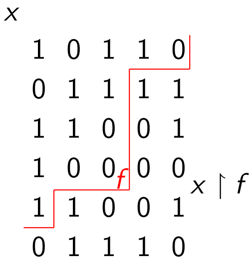

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