Two-cardinal combinatorics, guessing models, and cardinal arithmetic

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This talk is about joint work with Šárka Stejskalová.

I. Introduction

Two-cardinal lists

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Suppose that κ is an uncountable cardinal.

- *(Jech, 1973) κ is strongly compact if and only if, for all $\lambda \geq \kappa$, every (κ, λ)-list has a cofinal branch.*
- *(Magidor, 1974) κ is supercompact if and only if, for all $\lambda \geq \kappa$, every (κ, λ)-list has an ineffable branch.*

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- *μ-slender* if for all sufficiently large θ, there is a club $C \subseteq \mathscr{P}_\kappa H(\theta)$ such that, for all $M \in C$ and all $y \in M \cap \mathscr{P}_\mu\lambda$, we have $d_{M\cap\lambda} \cap y \in M$.

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Theorem (Weiß, 2012)

If κ is supercompact, then, in the extension by the Mitchell forcing $\mathbb{M}(\omega, \kappa)$, ISP$(\omega_1, \omega_2, \geq\omega_2)$ holds.

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We are especially interested in removing the requirement of *ineffability* from the hypotheses.

II. Slender lists and almost guessing

Almost guessing

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Each of these principles ends up being equivalent to an analogously modified variation of $\mathsf{SP}(\ldots)$.

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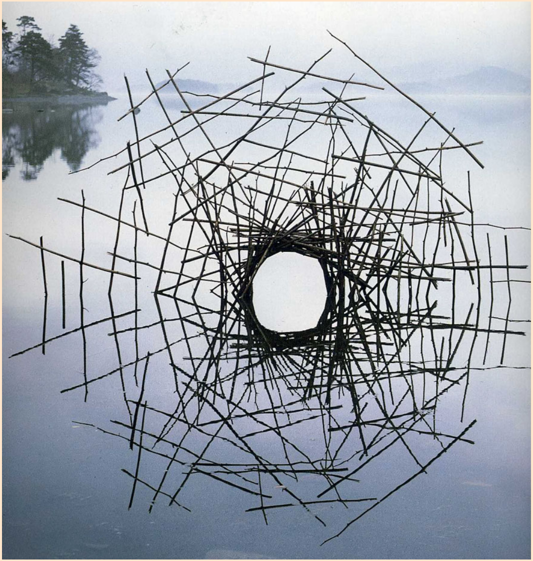
III. Cardinal Arithmetic



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This was previously known only under the additional assumption that $2^{|X|} < \lambda$.

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Observation (Viale)

Suppose that $\lambda > \chi = \mathrm{cf}(\lambda)$, $\mu^\chi < \lambda$ for all $\mu < \lambda$, and $\mathrm{CP}(\mathcal{D})$ holds for some covering matrix $\mathcal{D}$ for λ^+.

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Suppose that $\mathcal{D} = \langle D(i, \beta) \mid i < \chi,\ \beta < \lambda^+ \rangle$ is a covering matrix for λ^+. Then $\mathrm{CP}(\mathcal{D})$ is the assertion that there is an unbounded $A \subseteq \lambda^+$ such that, for every $x \in [A]^\chi$, there are $i < \chi$ and $\beta < \lambda^+$ such that $x \subseteq D(i, \beta)$.

Observation (Viale)

Suppose that $\lambda > \chi = \mathrm{cf}(\lambda)$, $\mu^\chi < \lambda$ for all $\mu < \lambda$, and $\mathrm{CP}(\mathcal{D})$ holds for some covering matrix $\mathcal{D}$ for λ^+. Then $\lambda^\chi = \lambda^+$.

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In particular, this holds under $\mathrm{ISP}(\omega_1, \omega_2, {\geq}\omega_2)$.

Narrow systems and SCH

In light of the fact that $\mathsf{ISP}(\omega_2, \omega_2, {\geq}\omega_2)$ (or even, as we have seen, a variation of $\mathsf{SP}(\omega_2, \omega_2, {\geq}\omega_2)$) implies SCH, it is natural to ask whether, e.g., (I)$\mathsf{TP}(\omega_2, {\geq}\omega_2)$ does the same.

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The notion of *narrow system* can be generalized to the $\mathscr{P}_\kappa\lambda$ setting.

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Theorem

Suppose that there is a proper class of supercompact cardinals. Then there is a class forcing extension in which $\mathsf{NSP}(\mathscr{P}_\kappa\lambda)$ *holds for all uncountable* $\kappa \leq \lambda$ *with* κ *regular.*

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Theorem

Suppose that $\chi < \chi^+ < \kappa < \lambda$ *are cardinals such that* $\mathrm{cf}(\lambda) = \chi$*,* κ *is regular, and* $\mathsf{NSP}(\mathscr{P}_\kappa \lambda^+)$ *holds.*

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Suppose that $\chi < \chi^+ < \kappa < \lambda$ *are cardinals such that* $\mathrm{cf}(\lambda) = \chi$*,* κ *is regular, and* $\mathsf{NSP}(\mathscr{P}_\kappa \lambda^+)$ *holds. Then* $\mathsf{CP}(\mathcal{D})$ *holds for every covering matrix* $\mathcal{D}$ *for* λ^+*.*

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Proof sketch.

Let $\mathcal{D} = \langle D(i, \beta) \mid i < \chi, \ \beta < \lambda^+ \rangle$ be a covering matrix for λ^+.

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Corollary

Suppose that $\kappa \geq \omega_2$ *is a regular cardinal and* $\mathsf{NSP}(\mathscr{P}_\kappa \lambda)$ *holds for all* $\lambda \geq \kappa$*.*

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Suppose that $\kappa \geq \omega_2$ *is a regular cardinal and* $\mathsf{NSP}(\mathscr{P}_\kappa \lambda)$ *holds for all* $\lambda \geq \kappa$. *Then* SCH *holds above* κ.

References

All new results in this talk come from three papers in preparation:

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All artwork by Andy Goldsworthy.

Thank you for your attention.