Ramsey theory over Partitions

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References

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Strong Colorings

- Strong colorings, which are witnesses to negative Ramsey relations, possess fantastic combinatorial properties that are used in a variety of arguments — among them arguments for producing stronger strong colorings from given ones.
- The first point of this talk is that strong colorings can be harnessed also to obtaining positive Ramsey relations on uncountable cardinals in some models of ZFC.
- The second is that in other models of ZFC there are even stronger strong colorings than the ones known so far.

Notation

- $A \subseteq \kappa$ is c-homogeneous for a coloring $c : [\kappa]^2 \to \theta$ if $c \restriction [A]^2$ is constant;
- it is c-omnichromatic if $\mathrm{Im}(c \restriction [A]^2) = \theta$;
- the negation of c-omnichromatic is c-weak.

- $\kappa \longrightarrow (\lambda)^2_\theta$ says that for *every* coloring $c : [\kappa]^2 \to \theta$ there is a c-homogeneous $A \in [\kappa]^\lambda$
- $\kappa \to [\lambda]^2_\theta$ says that for every c as above there is $A \in [\kappa]^\lambda$ which is c-weak.
- The negation $\kappa \nrightarrow [\lambda]^2_\theta$ holds if there is a coloring $c : [\kappa]^2 \to \theta$ such that *all* $A \in [\kappa]^\lambda$ are c-omnichromatic.

Such a c are called *strong*.

Ramsey Relations

- $\aleph_0 \to (\aleph_0)^2_n$ for all $n < \omega$, in particular $\aleph_0 \to [\aleph_0]^2_{\aleph_0}$,
- $\lambda^+ \not\to [\lambda^+]^2_{\lambda^+}$ for all regular λ: there are strong $c : [\lambda^+]^2 \to \lambda^+$.

Stronger strong colorings

- Parameters: number of colors, broader class of Target sets, higher dimension, more patterns.
- $\mathrm{Pr}_0(\aleph_1, \frac{\aleph_0 \circledast \aleph_1}{1 \circledast \aleph_1}, \aleph_1, \aleph_0)$ $\exists\, c : [\omega_1]^2 \to \omega_1$ s.t $\forall k, l < \omega$

$\forall \mathcal{A} \subseteq [\omega_1]^k$, $|\mathcal{A}| = \aleph_0$ $\forall \mathcal{B} \subseteq [\omega_1]^l$ $|\mathcal{B}| = \aleph_1$, Both $\mathcal{A}, \mathcal{B}$ pairwise disjoint

$\exists\, a \in \mathcal{A}$ $\forall (\alpha_{i,j})_{i<k, j<l} \subseteq \omega_1$ $\exists\, b \in \mathcal{B}$ $\max a < \min b$

and $\bigwedge_{i<k, j<l} c(a(i), b(j)) = \alpha_{i,j}$ where $a(i), b(j)$ are i-th and j-th members of a, b resp.

- Todorcevic proved that $\mathrm{Pr}_0(\aleph_1, \frac{\aleph_0 \circledast \aleph_1}{1 \circledast \aleph_1}, \aleph_1, \aleph_0)$ follows from a strongly Luzin set.

Relaxing Homogeneity to Relative Homogeneity

- When facing strong opponents it is good practice to Divide and Conquer. Let us fix some partition $p : [\kappa]^2 \to \mu$.
- Given a coloring $c : [\kappa]^2 \to \theta$ let us say that $A \subseteq \kappa$ is (p, c)-homogeneous [(p, c)-weak] if for every p-cell $X = p^{-1}(\{j\})$, $j < \mu$, it holds that $c \restriction (X \cap [A]^2)$ is constant [is c-weak].
- If for every coloring $c : [\kappa]^2 \to \theta$ there exists a (p, c)-homogeneous [(p, c)-weak] $A \in [\kappa]^\lambda$ then $\kappa \to_p (\lambda)^2_\theta$ [$\kappa \to_p [\lambda]^2_\theta$] holds;
- if, on the other hand, there is a c such that every $A \in [\kappa]^\lambda$, for some p-cell X the set $[A]^2 \cap X$ is omnichromatic, then c witnesses $\kappa \not\to_p [\lambda]^2_\theta$.

Stronger Strong colorings over p

Let τ denote assignments of colors to cells: $\tau : \mu \to \theta$.

- $\kappa \to_p (\lambda)^2_\theta \iff (\forall c)(\exists A)(\exists \tau)(\forall \{\alpha, \beta\} \in [A]^2)\, c(\alpha, \beta) = \tau(p(\alpha, \beta))$
- $\kappa \to_p [\lambda]^2_\theta \iff (\forall c)(\exists A)(\exists \tau)(\forall \{\alpha, \beta\} \in [A]^2)\, c(\alpha, \beta) \neq \tau(p(\alpha, \beta))$
- $\kappa \nrightarrow_p [\lambda]^2_\theta \iff (\exists c)(\forall A)(\forall \tau)(\exists \{\alpha, \beta\} \in [A]^2)\, c(\alpha, \beta) = \tau(p(\alpha, \beta))$

So $\mathrm{Pr}_0(\aleph_1, \frac{\aleph_0 \circledast \aleph_1}{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ means: replace a matrix of colors by a matrix of assignments $\tau_{i,j}$.

$$\bigwedge_{i,j} c(a(i), b(j)) = \tau_{i,j}(p(a(i), b(j)))$$

The strongest strong colorings may exists over every countable partition

If the CH holds or after adding at least $\aleph_2$ Cohen reals,

$$\mathrm{Pr}_0(\aleph_1, \frac{\aleph_0 \circledast \aleph_1}{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$$

holds for *every* countable $p : [\omega_1]^2 \to \omega_0$.

Theorem [KRS3]. $\mathrm{Pr}_0(\aleph_1, \frac{\aleph_0 \circledast \aleph_1}{1 \circledast \aleph_1}, \aleph_1, \aleph_0)_p$ follows for all ℓ_∞-coherent partitions from just a non-meager set of reals of cardinality $\aleph_1$. Actually equivalent to $\mathrm{non}(\mathcal{M}) = \aleph_1$. It does not follow for *all* countable partitions from a Luzin set together with a Souslin tree.

$p : [\omega_1]^2 \to \omega_0$ is ℓ_∞-coherent if $\{p(\alpha, \beta) - p(\alpha, \beta') \mid \alpha < \beta\}$ is bounded for all $\beta < \beta' < \omega_1$.

Stretching the number of colors from λ to λ^+.

Theorem [KRS2]. Let λ be an infinite cardinal. For every coloring $c : [\lambda]^+ \to \lambda$ there is a coloring $c^+ : [\lambda^+]^2 \to \lambda^+$ such that for every $\chi \leq \mathrm{cf}(\lambda)$ and $p : [\lambda^+]^2 \to \mu$ for some $\mu \leq \lambda$:

- If c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda, \chi)_p$ then c^+ witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^+, \lambda^+, \chi)_p$.
- If c witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^{\circledast}\lambda^+, \lambda, \chi)_p$ then c^+ witnesses $\mathrm{Pr}_1(\lambda^+, \lambda^{\circledast}\lambda^+, \lambda^+, \chi)_p$.

Positive relations from Forcing Axioms

1. Theorem [KRS1]. If $MA_{\aleph_1}(K)$ then for some $p : [\aleph_1]^2 \to \aleph_0$ it holds that $\aleph_1 \to_p (\aleph_1)^2_{\aleph_0}$.
2. Theorem [CRS1]. If $\lambda = \lambda^{<\lambda}$ and the Generalized MA holds at λ^+ then for every $(< \lambda)$-saturated partition with injective and almost disjoint fibres, for every coloring $c : [\lambda]^+ \to \lambda$ there is a decomposition $\dot\bigcup_{j<\lambda} X_j$ in which each X_j is of size λ^+, $[X_j]^2$ meets all p-cells and X_j is (p, c)-homogeneous.

Characterizing positivity over p

- Theorem [CRS1]. Assume $\mathrm{MA}_{\aleph_1}$. Let $p : [\omega_1]^2 \to \omega_0$ be a countable partition. Then $\aleph_1 \to_p [\aleph_1]^2_{\aleph_0,\mathrm{finite}}$ holds iff for some uncountable $X \subseteq \omega_1$ the restriction $p \restriction [X]^2$ witnesses the strong coloring principle $\mathrm{U}(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$ by Lambie-Hanson and Rinot.

$\mathrm{U}(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$ holds iff there exists $c : [\omega_1]^2 \to \omega_0$ such that for every pairwise disjoint, uncountable family $\mathcal{A}$ of finite subsets of ω_1 and $n < \omega$ there exists an uncountable $\mathcal{B} \subseteq \mathcal{A}$ such that $\min\{p(\alpha, \beta) \mid (\alpha, \beta) \in a \times b\} > n$ for all $a < b$ in $\mathcal{B}$. Unlike the stronger $\mathrm{Pr}_1(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$, the principle $\mathrm{U}(\aleph_1, \aleph_1, \aleph_0, \aleph_0)$ holds in ZFC.

A miniature

- Sierpinski showed that $2^{\aleph_0} \not\to [\aleph_1]^2_2$.
- Shelah proved the consistency (relative to a measurable cardinal) of $2^{\aleph_0} \to (\aleph_1)^2_3$ in "Was Sierpinski right I". Omitting one out of two colors is impossible, but omitting one out of three colors may be possible.
- Assume $2^{\aleph_0} \to [\aleph_1]^2_{4,2}$, that we can omit two out of four colors by passing to a set of reals of size $\aleph_1$. (Will hold it two of Shela's problems from WSRI hold together: $2^{\aleph_0} \to [\aleph_2]^2_3$ and $\aleph_2 \to [\aleph_1]^2_3$).

Let p be some Sierpinski coloring. Then $2^{\aleph_0} \to_p (\aleph_1)^2_2$.