How small can the first measurable cardinal be?

Asaf Karagila

University of Leeds

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Joint works with Yair Hayut and with Jiachen Yuan

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Theorem (Jech 1968)

It is consistent with ZF*, relative to the existence of a measurable cardinal, that $\aleph_1$ is measurable.*

Thank you for your attention!

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(It is also not clear whether 2 and 3 are equivalent or not with the current tools.)

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If κ is critical, then it is reflects being weakly critical. That is, there is a measure-1 set of weakly critical cardinals below κ.

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This method extends to any other reflection-style proofs.

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We call this class a *symmetric extension*, and we say that $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is a *symmetric system*.

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Suppose $V \models \mathsf{ZF}$, κ a measurable cardinal in V, and U is a measure on κ. Let $W \supseteq V$ such that for every $A \subseteq \kappa$, $V[A]$ has a unique extension of U to a measure on κ.

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Proof.

Let U_A denote the unique extension in $V[A]$, and let $U^+ = \bigcup\{U_A \mid A \subseteq \kappa\}$.

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Proof.

Let U_A denote the unique extension in $V[A]$, and let $U^+ = \bigcup\{U_A \mid A \subseteq \kappa\}$. We claim that U^+ is a κ-complete ultrafilter.

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4. If $\gamma < \kappa$ and $\{A_\alpha \mid \alpha < \gamma\} \subseteq U^+$, code the family by some C, then $U_{A_\alpha} \subseteq U_C$ for all $\alpha < \gamma$. Moreover, $\{A_\alpha \mid \alpha < \gamma\} \in V[C]$, so its intersection is there as well. □

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We wish to emulate this sort of condition in the case of symmetric extensions.

Inaccurately Stated Theorem (Hayut–K.)

Suppose $j\colon V \to M$ is an elementary embedding and $\langle \mathbb{P}, \mathscr{G}, \mathscr{F} \rangle$ is a symmetric system. Then j lifts to the symmetric extension if the following conditions hold:

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The new change is significant, since it opens up the door for a lot of the interesting cases where $j(\mathbb{P})=\mathbb{P}\times\mathbb{Q}$ and we already have $H\in V$.

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General gist of a Proof.

Let κ be a measurable cardinal. Consider the Easton support product of $\mathbb{Q}_\alpha$ for $\alpha < \kappa$ inaccessible, which adds a non-reflecting stationary subset to S^α_ω. This partial order is homogeneous, so we can take the product of the automorphism groups acting pointwise on each $\mathbb{Q}_\alpha$, with $\mathscr{F}$ being the filter generated by $\prod_{\alpha<\kappa} H_\alpha$, where on a tail of α, $H_\alpha = \mathrm{Aut}(\mathbb{Q}_\alpha)$. □

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To see that κ remains Mahlo, note that if C is a club, then in $V[C]$, κ is Mahlo, so C must contain a strongly inaccessible cardinal. But as the forcing did not collapse any cardinals or change the continuum function, those remain strongly inaccessible in the symmetric extension. □

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So in K it must be that $o(\kappa) \geqslant 2$. □

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Question

Is there a reasonable construction, starting from much stronger large cardinal assumptions, of a model of ZF *in which the least inaccessible cardinal is the least measurable cardinal?*

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(And apologies for whatever I screwed up.)